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MARKSCHEME

May 2013

MATHEMATICS

Higher Level

Paper 2

15 pages

Instructions to Examiners

Abbreviations

M Marks awarded for attempting to use a correct **Method**; working must be seen.

(M) Marks awarded for **Method**; may be implied by **correct** subsequent working.

A Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.

(A) Marks awarded for an **Answer** or for **Accuracy**; may be implied by **correct** subsequent working.

R Marks awarded for clear **Reasoning**.

N Marks awarded for **correct** answers if **no** working shown.

AG Answer given in the question and so no marks are awarded.

Using the markscheme

1 General

Write the marks in red on candidates' scripts, in the right hand margin.

- Show the **breakdown** of individual marks awarded using the abbreviations ***M1***, ***A1***, *etc*.
- Write down the total for each **question** (at the end of the question) and **circle** it.

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) depend on the preceding ***M*** mark(s), if any.
- Where ***M*** and ***A*** marks are noted on the same line, *e.g.* ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (*e.g.* substitution into a formula) and ***A1*** for using the **correct** values.
- Where the markscheme specifies ***(M2)***, ***N3***, *etc.*, do **not** split the marks.
- Once a correct answer to a question or part-question is seen, ignore further working.

3 *N* marks

*Award **N** marks for **correct** answers where there is **no** working.*

- Do **not** award a mixture of ***N*** and other marks.
- There may be fewer ***N*** marks available than the total of ***M***, ***A*** and ***R*** marks; this is deliberate as it penalizes candidates for not following the instruction to show their working.

4 Implied marks

Implied marks appear in ***brackets e.g. (M1)****, and can only be awarded if* ***correct*** *work is seen or if implied in subsequent working.*

- Normally the correct work is seen or implied in the next line.
- Marks **without** brackets can only be awarded for work that is **seen**.

5 Follow through marks

*Follow through (****FT****) marks are awarded where an incorrect answer from one* ***part*** *of a question is used correctly in* ***subsequent*** *part(s). To award* ***FT*** *marks,* ***there must be working present*** *and not just a final answer based on an incorrect answer to a previous part.*

- If the question becomes much simpler because of an error then use discretion to award fewer ***FT*** marks.
- If the error leads to an inappropriate value (*e.g.* $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).
- Within a question part, once an error is made, no further **dependent** ***A*** marks can be awarded, but ***M*** marks may be awarded if appropriate.
- Exceptions to this rule will be explicitly noted on the markscheme.

6 Mis-read

*If a candidate incorrectly copies information from the question, this is a mis-read (****MR****). Apply a* ***MR*** *penalty of 1 mark to that question. Award the marks as usual and then write –1(****MR****) next to the total. Subtract 1 mark from the total for the question. A candidate should be penalized only once for a particular mis-read.*

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.
- If the ***MR*** leads to an inappropriate value (*e.g.* $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).

7 Discretionary marks *(d)*

An examiner uses discretion to award a mark on the rare occasions when the markscheme does not cover the work seen. The mark should be labelled ***(d)*** *and a brief* ***note*** *written next to the mark explaining this decision.*

8 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If in doubt, contact your team leader for advice.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc*.
- Alternative solutions for part-questions are indicated by **EITHER** . . . **OR**.
- Where possible, alignment will also be used to assist examiners in identifying where these alternatives start and finish.

9 Alternative forms

*Unless the question specifies otherwise, **accept** equivalent forms.*

- As this is an international examination, accept all alternative forms of **notation**.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, **simplified** answers, (which candidates often do not write in examinations), will generally appear in brackets. Marks should be awarded for either the form preceding the bracket or the form in brackets (if it is seen).

Example: for differentiating $f(x) = 2\sin(5x-3)$, the markscheme gives:

$$f'(x) = \left(2\cos(5x-3)\right)5 \quad \left(=10\cos(5x-3)\right) \qquad \textbf{\textit{A1}}$$

Award ***A1*** for $\left(2\cos(5x-3)\right)5$, even if $10\cos(5x-3)$ is not seen.

10 Accuracy of Answers

*The method of dealing with accuracy errors on a whole paper basis by means of the Accuracy Penalty **(AP)** no longer applies.*

Instructions to examiners about such numerical issues will be provided on a question by question basis within the framework of mathematical correctness, numerical understanding and contextual appropriateness.

*The rubric on the front page of each question paper is given for the guidance of candidates. The markscheme (MS) may contain instructions to examiners in the form of "Accept answers which round to n significant figures (sf)". Where candidates state answers, required by the question, to fewer than n sf, award **A0**. Some intermediate numerical answers may be required by the MS but not by the question. In these cases only award the mark(s) if the candidate states the answer exactly or to at least 2sf.*

11 Crossed out work

If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work.

12 More than one solution

Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise.

SECTION A

1. $\frac{5\times6+6k+7\times3+8\times1+9\times2+10\times1}{13+k}=6.5$ (or equivalent) ***(M1)(A1)(A1)***

Note: Award ***(M1)(A1)*** for correct numerator, and ***(A1)*** for correct denominator.

$0.5k=2.5\Rightarrow k=5$ ***A1***

[4 marks]

2. METHOD 1

determinant $=0$ ***M1***

$k(-2-16)-(0-12)+2(0+3)=0$ ***(M1)(A1)***

$-18k+18=0$ ***(A1)***

$k=1$ ***A1***

METHOD 2

writes in the form

$$\begin{pmatrix} k & 1 & 2 & 4 \\ 0 & -1 & 4 & 5 \\ 3 & 4 & 2 & 1 \end{pmatrix}$$ (or attempts to solve simultaneous equations) ***(M1)***

Having two 0's in first column (obtaining two equations in the same two variables) ***M1***

$$\begin{pmatrix} k & 1 & 2 & 4 \\ 0 & -1 & 4 & 5 \\ 0 & 0 & 18k-18 & 21k-27 \end{pmatrix}$$ (or isolating one variable) ***A1***

Note: The ***A1*** is to be awarded for the $18k-18$. The final column may not be seen.

$k=1$ ***(M1)A1***

[5 marks]

3. Let X represent the length of time a journey takes on a particular day.

(a) $P(X>15)=0.0912112819\ldots=0.0912$ ***(M1)A1***

(b) Use of correct Binomial distribution ***(M1)***

$N\sim B(5, 0.091...)$

$1-0.0912112819\ldots=0.9087887181\ldots$

$1-(0.9087887181\ldots)^5=0.380109935\ldots=0.380$ ***(M1)A1***

Note: Allow answers to be given as percentages.

[5 marks]

4. volume $= \pi \int x^2 \mathrm{d}y$ ***(M1)***

$x = \arcsin y + 1$ ***(M1)(A1)***

volume $= \pi \int_0^1 (\arcsin y + 1)^2 \mathrm{d}y$ ***A1***

Note: ***A1*** is for the limits, provided a correct integration of y.

$= 2.608993\ldots\pi = 8.20$ ***A2*** ***N5***

[6 marks]

5. $\frac{1}{2} r^2 \times 1 = 7$ ***M1***

$r = 3.7\ldots \left(= \sqrt{14}\right)$ (or 37… mm) ***(A1)***

height $= 2r\cos\left(\frac{\pi - 1}{2}\right)$ $\left(\text{or } 2r\sin\frac{1}{2}\right)$ ***(M1)(A1)***

3.59 or anything that rounds to 3.6 ***A1***

so the dimensions are 3.7 by 3.6 (cm or 37 by 36 mm) ***A1***

[6 marks]

6. other root is $2 - i$ ***(A1)***

a quadratic factor is therefore $(x - 2 + i)(x - 2 - i)$ ***(M1)***

$= x^2 - 4x + 5$ ***A1***

$x + 1$ is a factor ***A1***

$(x - 2)^2$ is a factor ***A1***

$p(x) = a(x + 1)(x - 2)^2(x^2 - 4x + 5)$ ***(M1)***

$p(0) = 4 \Rightarrow a = \frac{1}{5}$ ***A1***

$p(x) = \frac{1}{5}(x + 1)(x - 2)^2(x^2 - 4x + 5)$

[7 marks]

7. (a) let the distance the cable is laid along the seabed be y

$y^2 = x^2 + 200^2 - 2 \times x \times 200 \cos 60^0$ ***(M1)***

(or equivalent method)

$y^2 = x^2 - 200x + 40000$ ***(A1)***

$\text{cost} = C = 80y + 20x$ ***(M1)***

$C = 80(x^2 - 200x + 40000)^{\frac{1}{2}} + 20x$ ***A1***

[4 marks]

(b) $x = 55.2786\ldots = 55$ (m to the nearest metre) ***(A1)A1***

$\left(x = 100 - \sqrt{2000}\right)$

[2 marks]

Total [6 marks]

8. (a) the three girls can sit together in $3! = 6$ ways ***(A1)***

this leaves 4 'objects' to arrange so the number of ways this can be done is $4!$ ***(M1)***

so the number of arrangements is $6 \times 4! = 144$ ***A1***

[3 marks]

(b) Finding more than one position that the girls can sit ***(M1)***

Counting exactly four positions ***(A1)***

number of ways = $4 \times 3! \times 3! = 144$ ***M1A1*** ***N2***

[4 marks]

Total [7 marks]

9. (a) $\Delta = b^2 - 4ac = 4k^2 - 4 \times 3 \times (k-1) = 4k^2 - 12k + 12$ ***M1A1***

Note: Award ***M1A1*** if expression seen within quadratic formula.

EITHER

$144 - 4 \times 4 \times 12 < 0$ ***M1***

Δ always positive, therefore the equation always has two distinct real roots ***R1***

(and cannot be always negative as $a > 0$)

OR

sketch of $y = 4k^2 - 12k + 12$ or $y = k^2 - 3k + 3$ not crossing the x-axis ***M1***

Δ always positive, therefore the equation always has two distinct real roots ***R1***

OR

write Δ as $4(k-1.5)^2 + 3$ ***M1***

Δ always positive, therefore the equation always has two distinct real roots ***R1***

[4 marks]

continued ...

Question 9 continued

(b) closest together when Δ is least **(M1)**
minimum value occurs when $k = 1.5$ **(M1)A1**

[3 marks]

Total [7 marks]

10. (a) $X \sim \text{Po}(0.25\text{T})$ **(A1)**
Attempt to solve $\text{P}(X \le 3) = 0.6$ **(M1)**
$T = 12.8453\ldots = 13$ (minutes) **A1**

Note: Award ***A1M1A0*** if *T* found correctly but not stated to the nearest minute.

[3 marks]

(b) let X_1 be the number of cars that arrive during the first interval and X_2 be the number arriving during the second.
X_1 and X_2 are $\text{Po}(2.5)$ **(A1)**
P (all get on) $= \text{P}(X_1 \le 3) \times \text{P}(X_2 \le 3) + \text{P}(X_1 = 4) \times \text{P}(X_2 \le 2)$
$+\text{P}(X_1 = 5) \times \text{P}(X_2 \le 1) + \text{P}(X_1 = 6) \times \text{P}(X_2 = 0)$ **(M1)**
$= 0.573922\ldots + 0.072654\ldots + 0.019192\ldots + 0.002285\ldots$ **(M1)**
$= 0.668\ (053\ldots)$ **A1**

[4 marks]

Total [7 marks]

SECTION B

11. (a) $\vec{PQ} = \begin{pmatrix} 8 \\ 6 \\ 4 \end{pmatrix}$ ***(A1)***

equation of line: $\boldsymbol{r} = \begin{pmatrix} -3 \\ -1 \\ 2 \end{pmatrix} + t\begin{pmatrix} 8 \\ 6 \\ 4 \end{pmatrix}$ (or equivalent) ***M1A1***

> **Note**: Award ***M1A0*** if $\boldsymbol{r} =$ is omitted.

[3 marks]

(b) **METHOD 1**

$x: \quad -4 + 5s = -3 + 8t$
$y: \quad 2s = -1 + 6t$
$z: \quad 4 = 2 + 4t$ ***M1***

solving any two simultaneously ***M1***

$t = 0.5,\ s = 1$ (or equivalent) ***A1***

verification that these values give R when substituted into **both** equations
(or that the three equations are consistent and that one gives R) ***R1***

METHOD 2

$(1, 2, 4)$ is given by $t = 0.5$ for L_1 and $s = 1$ for L_2 ***M1A1A1***

because $(1, 2, 4)$ is on both lines it is the point of intersection of the two lines ***R1***

[4 marks]

(c) $\begin{pmatrix} 5 \\ 2 \\ 0 \end{pmatrix} \bullet \begin{pmatrix} 4 \\ 3 \\ 2 \end{pmatrix} = 26 = \sqrt{29} \times \sqrt{29} \cos\theta$ ***M1***

$\cos\theta = \dfrac{26}{29}$ ***(A1)***

$\theta = 0.459$ or 26.3° ***A1***

[3 marks]

continued …

Question 11 continued

(d) $\vec{RP} = \begin{pmatrix} -3 \\ -1 \\ 2 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} = \begin{pmatrix} -4 \\ -3 \\ -2 \end{pmatrix}, \left|\vec{RP}\right| = \sqrt{29}$ ***(M1)A1***

Note: This could also be obtained from $\left\| 0.5 \begin{pmatrix} 8 \\ 6 \\ 4 \end{pmatrix} \right\|$

EITHER

$\vec{RS_1} = \begin{pmatrix} -4 \\ 0 \\ 4 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} = \begin{pmatrix} -5 \\ -2 \\ 0 \end{pmatrix}, \left|\vec{RS_1}\right| = \sqrt{29}$ ***A1***

$\therefore \vec{OS_2} = \vec{OS_1} + 2\vec{S_1R} = \begin{pmatrix} -4 \\ 0 \\ 4 \end{pmatrix} + 2\begin{pmatrix} 5 \\ 2 \\ 0 \end{pmatrix}$ ***M1A1***

$\left(\text{or } \vec{OS_2} = \vec{OR} + \vec{S_1R} = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} + \begin{pmatrix} 5 \\ 2 \\ 0 \end{pmatrix} \right)$

$= \begin{pmatrix} 6 \\ 4 \\ 4 \end{pmatrix}$

S_2 is $(6, 4, 4)$ ***A1***

OR

$\begin{pmatrix} -4+5s \\ 2s \\ 4 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 5s-5 \\ 2s-2 \\ 0 \end{pmatrix}$ ***M1***

$(5s-5)^2 + (2s-2)^2 = 29$ ***M1A1***

$29s^2 - 58s + 29 = 29$

$s(s-2) = 0,\ s = 0, 2$

$(6, 4, 4)$ (and $(-4, 0, 4)$) ***A1***

Note: There are several geometrical arguments possible using information obtained in previous parts, depending on what forms the previous answers had been given.

[6 marks]

(e) **EITHER**

midpoint of $[PS_1]$ is $M(-3.5, -0.5, 3)$ ***M1A1***

$$\vec{RM} = \begin{pmatrix} -4.5 \\ -2.5 \\ -1 \end{pmatrix}$$ ***A1***

OR

$$\vec{RS_1} = \begin{pmatrix} -5 \\ -2 \\ 0 \end{pmatrix}$$ ***M1***

the direction of the line is $\vec{RS_1} + \vec{RP}$

$$\begin{pmatrix} -5 \\ -2 \\ 0 \end{pmatrix} + \begin{pmatrix} -4 \\ -3 \\ -2 \end{pmatrix} = \begin{pmatrix} -9 \\ -5 \\ -2 \end{pmatrix}$$ ***M1A1***

THEN

the equation of the line is:

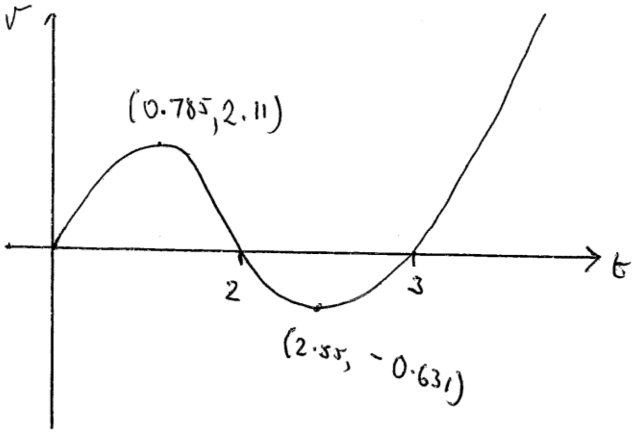
$$\boldsymbol{r} = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} + t\begin{pmatrix} 9 \\ 5 \\ 2 \end{pmatrix}$$ or equivalent ***A1***

Note: Marks cannot be awarded for methods involving halving the angle, unless it is clear that the candidate considers also the equation of the plane of L_1 and L_2 to reduce the number of parameters involved to one (to obtain the vector equation of the required line).

[4 marks]

Total [20 marks]

12. (a)



A1A1A1

Note: Award ***A1*** for general shape, ***A1*** for correct maximum and minimum, ***A1*** for intercepts.

Note: Follow through applies to (b) and (c).

[3 marks]

(b) $0 \le t < 0.785, \left(\text{or } 0 \le t < \dfrac{5-\sqrt{7}}{3}\right)$ ***A1***

(allow $t < 0.785$)

and $t > 2.55 \left(\text{or } t > \dfrac{5+\sqrt{7}}{3}\right)$ ***A1***

[2 marks]

(c) $0 \le t < 0.785, \left(\text{or } 0 \le t < \dfrac{5-\sqrt{7}}{3}\right)$ ***A1***

(allow $t < 0.785$)

$2 < t < 2.55, \left(\text{or } 2 < t < \dfrac{5+\sqrt{7}}{3}\right)$ ***A1***

$t > 3$ ***A1***

[3 marks]

(d) position of A: $x_A = \int t^3 - 5t^2 + 6t \text{ d}t$ ***(M1)***

$x_A = \dfrac{1}{4}t^4 - \dfrac{5}{3}t^3 + 3t^2 \quad (+c)$ ***A1***

when $t = 0$, $x_A = 0$ so $c = 0$ ***R1***

[3 marks]

continued …

Question 12 continued

(e) $\frac{dv_B}{dt} = -2v_B \Rightarrow \int \frac{1}{v_B} dv_B = \int -2dt$ ***(M1)***

$\ln|v_B| = -2t + c$ ***(A1)***

$v_B = Ae^{-2t}$ ***(M1)***

$v_B = -20$ when $t = 0$ so $v_B = -20e^{-2t}$ ***A1***

[4 marks]

(f) $x_B = 10e^{-2t}(+c)$ ***(M1)(A1)***

$x_B = 20$ when $t = 0$ so $x_B = 10e^{-2t} + 10$ ***(M1)A1***

meet when $\frac{1}{4}t^4 - \frac{5}{3}t^3 + 3t^2 = 10e^{-2t} + 10$ ***(M1)***

$t = 4.41(290\ldots)$ ***A1***

[6 marks]

Total: [21 marks]

13. (a) $f(2) = 9$ ***(A1)***

$f^{-1}(x) = (x-1)^{\frac{1}{3}}$ ***A1***

$(f^{-1})'(x) = \frac{1}{3}(x-1)^{-\frac{2}{3}}$ ***(M1)***

$(f^{-1})'(9) = \frac{1}{12}$ ***A1***

$f'(x) = 3x^2$ ***(M1)***

$\frac{1}{f'(2)} = \frac{1}{3 \times 4} = \frac{1}{12}$ ***A1***

Note: The last ***M1*** and ***A1*** are independent of previous marks.

[6 marks]

(b) $g'(x) = e^{x^2} + 2x^2e^{x^2}$ ***M1A1***

$g'(x) > 0$ as each part is positive ***R1***

[3 marks]

continued …

Question 13 continued

(c) to find the x-coordinate on $y = g(x)$ solve

$2 = xe^{x^2}$ ***(M1)***

$x = 0.89605022078\ldots$ ***(A1)***

gradient $= (g^{-1})'(2) = \dfrac{1}{g'(0.896\ldots)}$ ***(M1)***

$= \dfrac{1}{e^{(0.896\ldots)^2}\left(1 + 2\times(0.896\ldots)^2\right)} = 0.172$ to 3sf ***A1***

(using the $\dfrac{dy}{dx}$ function on gdc $g'(0.896\ldots) = 5.7716028\ldots$

$\dfrac{1}{g'(0.896\ldots)} = 0.173$)

[4 marks]

(d) (i) $(x^3+1)e^{(x^3+1)^2} = 2$ ***A1***

$x = -0.470191\ldots$ ***A1***

(ii) **METHOD 1**

$(g \circ f)'(x) = 3x^2e^{(x^3+1)^2}\left(2(x^3+1)^2+1\right)$ ***(M1)(A1)***

$(g \circ f)'(-0.470191\ldots) = 3.85755\ldots$ ***(A1)***

$h'(2) = \dfrac{1}{3.85755\ldots} = 0.259\ (232\ldots)$ ***A1***

Note: The solution can be found without the student obtaining the explicit form of the composite function.

METHOD 2

$h(x) = (f^{-1} \circ g^{-1})(x)$ ***A1***

$h'(x) = (f^{-1})'\left(g^{-1}(x)\right) \times (g^{-1})'(x)$ ***M1***

$= \dfrac{1}{3}\left(g^{-1}(x) - 1\right)^{-\frac{2}{3}} \times (g^{-1})'(x)$ ***M1***

$h'(2) = \dfrac{1}{3}\left(g^{-1}(2) - 1\right)^{-\frac{2}{3}} \times (g^{-1})'(2)$

$= \dfrac{1}{3}(0.89605\ldots - 1)^{-\frac{2}{3}} \times 0.171933\ldots$

$= 0.259\ (232\ldots)$ ***A1*** ***N4***

[6 marks]

Total [19 marks]